

A new recurrence formula for generic exceptional orthogonal polynomials

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Abstract

A new recurrence relation for exceptional orthogonal polynomials is proposed, which holds for type 1, 2 and 3. As concrete examples, the recurrence relations are given for X_j -Hermite, Laguerre and Jacobi polynomials in $j = 1, 2$ case.

Keywords: exceptional orthogonal polynomials, Darboux transformation, recurrence relations

Mathematics Subject Classification: 33C45, 33C47, 42C05

1 Introduction

The exceptional orthogonal polynomials (XOPs) were introduced by Gómez-Ullate, Kamran and Milson as a generalization of the classical orthogonal polynomials (COPs), where the polynomials of the first several degree are missed[8]. After their introduction, the XOPs have been widely accepted and have been developed by many researchers. In particular, Sasaki and Odake have figured out the direct relationship between XOPs and COPs, and then derived many exceptional extensions of COPs belonging to the Askey scheme[14]. They have also introduced the multi-indexed orthogonal polynomials by using the repeated Darboux transformations of the XOPs[15].

In this article we will discuss the recurrence relations, one of the fundamental properties, of the XOPs. The XOPs satisfy the second-order differential (or difference) equation by construction, although they do not hold three-term recurrence relations which ordinary OPs are supposed to. Concerning the XOPs derived from the one-time Darboux transformation of the COPs, to the best of our knowledge, the shortest recurrence relations are $4j + 1$ -term recurrence relations:

$$p_j^2(x)Q_n(x) = \sum_{k=n-2j}^{n+2j} \alpha_{n,k} Q_k(x),$$

where all coefficients $c_k^{(n)}$ do not depend on x . Remark that, if the coefficients of recurrence relations allow to depend on x , then there exist 5-term recurrence relations.

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The purpose of this article is to present the new recurrence relations of the form:

$$\left(\int p_j(x)dx\right)Q_n(x) = \sum_{k=n-j-1}^{n+j+1} \beta_{n,k}Q_k(x),$$

which form $2j + 3$ -term and hold for any type of XOPs.

This article is organized as follows. In §2, we review how to construct XOPs by using Darboux transformations [1, 3, 4]. In §3, we show the new recurrence formula for XOPs by observing the structure of the Darboux transformation. In §4, we give the concrete examples of the obtained recurrence relations especially for classical XOPs (Hermite, Laguerre and Jacobi). Finally we give brief summary of this article.

2 The exceptional orthogonal polynomials

The exceptional orthogonal polynomials can be characterized by Darboux transformations among the quasi-polynomial eigenfunctions of the classical Bôchner type operator[9, 16]. Those are the natural extensions of the COPs characterized by pure polynomial eigenfunctions of the classical Bôchner type operator. In this section, we briefly review how to construct the exceptional orthogonal polynomials[9, 14].

2.1 Classical orthogonal polynomials

We first explain COPs shortly before XOPs (see e.g. [12] for further details). COPs $\{P_n(x)\}_{n=0}^{\infty}$ are usually defined as polynomial solutions to the following Sturm-Liouville problem:

$$(A(x)\partial^2 + B(x)\partial + C(x))P_n = \lambda_n P_n, \quad \deg P_n = n, \quad (1)$$

or equivalently

$$w(x)^{-1} (A(x)w(x)P'_n(x))' = \lambda_n P_n(x), \quad (2)$$

where $\partial \equiv \frac{d}{dx}$ and

$$(A(x)w(x))' = B(x)w(x). \quad (3)$$

In order that (1) admits the polynomial sequence solution, $A(x), B(x), C(x)$ are supposed to be

$$\deg(A(x)) \leq 2, \quad \deg(B(x)) = 1, \quad \deg(C(x)) = 0 \quad (4)$$

and the polynomial solutions are found to be orthogonal polynomials whose orthogonality is described by

$$\int_D P_m(x)P_n(x)w(x)dx = h_n\delta_{mn} \quad (h_n \neq 0), \quad (5)$$

where $D \subset \mathbb{R} \cup \{\pm\infty\}$ is the corresponding interval s.t. $A(x)w(x)|_{x \in \partial D} = 0$. One of the specific feature of the COPs is that the derivatives of the COPs $\{P'_n(x)\}_{n=1}^{\infty}$ are again COPs whose orthogonality is given by

$$\int_D P'_m(x)P'_n(x)A(x)w(x)dx = \tilde{h}_n\delta_{mn} \quad (\tilde{h}_n \neq 0). \quad (6)$$

As is well-known, Böchner classified the all polynomials for (1) and showed up to Affine transformation the polynomial eigenfunctions belong to the classical orthogonal polynomials of Jacobi, Laguerre, Hermite or the Bessel polynomials[2]. Throughout this paper, we call the operator $A(x)\partial^2 + B(x)\partial + C(x)$ with (4) as Böchner type operator.

2.2 Quasi-polynomial eigenfunctions

As is mentioned before, XOPs can be characterized by Darboux transformations among the quasi-polynomial eigenfunctions of the classical Böchner type operator. Hence we shall examine all the possible quasi-polynomial eigenfunctions[17].

Let \mathbb{L} be the set of Böchner type operators defined by

$$\mathbb{L} = \{ \alpha_2 \partial^2 + \alpha_1 \partial + \alpha_0 \mid \alpha_1, \alpha_2 \in \mathbb{C}[x], 0 \leq \deg(\alpha_2) \leq 2, \deg(\alpha_1) = 1, \alpha_0 \in \mathbb{C} \}. \quad (7)$$

We consider a special class of eigenfunctions $\phi(x)$ of $\mathcal{L} \in \mathbb{L}$ which can be separated into a gauge part $\xi(x)$ and a polynomial part $p(x)$, that is

$$\phi(x) = \xi(x)p(x).$$

In the case of when $\xi'(x)/\xi(x)$ comes to be a rational function, we call $\phi(x) = \xi(x)p(x)$ a *quasi-polynomial* eigenfunction.

The gauge part of quasi-polynomial eigenfunctions can be determined by

$$\xi'(x; A, B, \gamma) / \xi(x; A, B, \gamma) = \eta(x, A, B, \gamma), \quad (8)$$

where

$$\eta(x; A, B, \gamma) = \frac{1}{2\pi i} \oint_{\gamma} \frac{B(z) - A'(z)}{A(z)} \frac{dz}{z - x}, \quad (9)$$

and γ is a positively oriented closed curve in $\mathbb{C} \setminus \{\text{zeros of } A(z), x\}$ which does not enclose the point $x \in \mathbb{C}$. Then it holds that

$$\xi^{-1} \circ \mathcal{L} \circ \xi \in \mathbb{L}, \quad (10)$$

where $\xi(x; A, B, \gamma)$ is formally given by

$$\xi_{\gamma}(x) = \xi(x; A, B, \gamma) = \exp \left(\int \eta(x; A, B, \gamma) dx \right).$$

Hereafter we employ the notation $\xi_{\gamma}(x) = \xi(x; A, B, \gamma)$, $\eta_{\gamma}(x) = \eta(x; A, B, \gamma)$ for simplicity. For later usage, let us introduce the function $A_{\gamma}(x)$ given by

$$A'_{\gamma}(x) / A_{\gamma}(x) = -\eta(x; A, 0, \gamma). \quad (11)$$

After fixing the gauge part of quasi-polynomial eigenfunctions, the polynomial part $p(x)$ of any degree n in x can be determined by solving the eigenvalue problem

$$(\xi^{-1} \circ \mathcal{L} \circ \xi)[p](x) = \lambda p(x),$$

up to the scaling constant, if $nA''/2 + B' + 2(A\xi'/\xi)' \neq 0$ for $n \in \mathbb{Z}_{\geq 0}$. Thus we may denote the n th degree monic polynomial eigenfunction of $\xi_\gamma^{-1} \circ \mathcal{L} \circ \xi_\gamma = \mathcal{L} + 2A\eta_\gamma \partial \in \mathbb{L}$ by $p_{\gamma,n}(x)$, that is

$$(\mathcal{L} + 2A\eta_\gamma \partial)[p_{\gamma,n}(x)] = \lambda_{\gamma,n} p_{\gamma,n}(x). \quad (12)$$

Let \mathbb{G} be the set of representative closed curves which determine the gauge part $\xi(x)$ of quasi-polynomial eigenfunctions of \mathcal{L} . Then we introduce the set of quasi-polynomial eigenfunctions of \mathcal{L} by

$$\Phi = \{\phi_{\gamma,n}(x) = \xi_\gamma(x) p_{\gamma,n}(x) \mid (\gamma, n) \in \mathbb{G} \times \mathbb{Z}_{\geq 0}\}. \quad (13)$$

Remark. In the case when $\gamma = \emptyset$, then ξ_\emptyset becomes a constant and therefore $\phi_{\emptyset,n}(x)$ belongs to COPs.

Since $\deg(A) \leq 2$ and $\deg(B) = 1$, we can find all gauge factors to be classified into at most four kinds depending on the choice of contour $\gamma \in \mathbb{G}$. Let \mathbf{Q} be the set of all poles of the integrand $(B(z) - A'(z))A(z)^{-1}(z-x)^{-1}$ in $\mathbb{C} \cup \{\infty\}$. The number of elements of \mathbf{Q} , denoted by $\#\mathbf{Q}$, is taken from 0 to 3, since the degree of A is less than or equal to 2. One can find that the integrand is identically zero if $\#\mathbf{Q} = 0$ and also $\mathbf{Q} = \{x\}$ if $\#\mathbf{Q} = 1$.

For any $\mathcal{L} \in \mathbb{L}$ we have two types of curves, \emptyset and III as follows:

- $\emptyset \in \mathbb{G}$ does not enclose any points in \mathbf{Q} ,
- III $\in \mathbb{G}$ encloses all points but x in \mathbf{Q} .

Note that, if $\#\mathbf{Q}$ is equal to 0 or 1, then the curves \emptyset and III can be treated as being the same curve. Moreover, if the set $\mathbf{Q} \setminus \{x\}$ contains two elements, say a_1 and a_2 , then two more additional representative closed curves I and II can be introduced as

- I $\in \mathbb{G}$ encloses a_1 , but not a_2 or x ,
- II $\in \mathbb{G}$ encloses a_2 , but not a_1 or x .

We obtain the type 1 and type 2 XOPs by taking the quasi-polynomials derived from the curves I and II as a seed function of the Darboux transformation discussed in the next subsection. The type 3 XOPs is derived from the quasi-polynomials whose gauge part is given by $\xi_{\text{III}}(x)[6]$.

Lemma 2.1. Let $\text{Res}_\zeta = \text{Res}_{z=\zeta} [(B(z) - A'(z))/(A(z)(z-x))]$ be a residue at the point $\zeta \in \mathbb{C} \cup \{\infty\}$.

For each $\mathcal{L} \in \mathbb{L}$, corresponding to the choice of the closed curve $\gamma \in \mathbb{G}$, we have the following types of functions $\eta_\gamma(x)$:

- For any pair of nonzero polynomial A of degree at most 2, and linear polynomial B ,
 - (i) $\eta_\emptyset = 0$,
 - (ii) $\eta_{\text{III}}(x) = -\text{Res}_x = (A'(x) - B(x))/A(x)$.

Only the following two cases admit additional two types of η :

- For any pair of polynomials A, B such that $A = a_0(x-a_1)(x-a_2) \neq 0$ with $a_1 \neq a_2$, $\deg B = 1$, and $B - A'$ has no common root with A ,
 - (iii) $\eta_{\text{I}} = \text{Res}_{a_1}$,
 - (iv) $\eta_{\text{II}} = \text{Res}_{a_2}$.

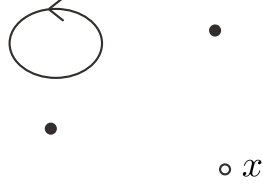


Figure 1: Case $\#Q = 3$, curve \emptyset

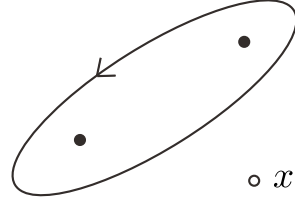


Figure 2: Case $\#Q = 3$, curve III

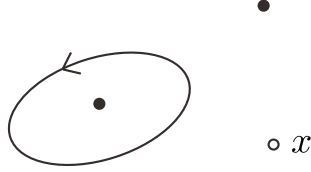


Figure 3: Case $\#Q = 3$, curve I

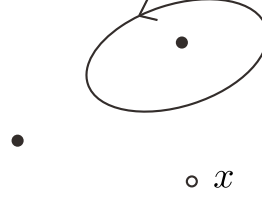


Figure 4: Case $\#Q = 3$, curve II

- For any pair of polynomials A, B such that $A = a_0(x - a_1) \neq 0$, $\deg B = 1$, and $B - A'$ has no common root with A , then
 (iii)' $\eta_I = \text{Res}_{a_1}$, (iv)' $\eta_{II} = \text{Res}_\infty$.

Remark. The cases (iii) and (iv) occur in the Böchner type operator of Jacobi polynomials, and also the cases (iii)' and (iv)' in the one of Laguerre polynomials.

For a given closed curve $\gamma \in \mathbb{G}$, there exists $\gamma^* \in \mathbb{G}$ such that

$$\xi_{\gamma^*}(x) = \kappa(w(x)\xi_\gamma(x))^{-1}$$

with some nonzero constant κ . By virtue of the arbitrary scaling factor of ξ_γ , the nonzero constant κ can be set to $\kappa = 1$. In a similarly manner for A_γ , one can obtain $A_{\gamma^*}(x) = A(x)/A_\gamma(x)$. Here and hereafter we assume that

$$\xi_\emptyset(x) = A_\emptyset(x) = 1, \quad w(x) = 1/(\xi_\gamma(x)\xi_{\gamma^*}(x)), \quad A(x) = A_\gamma(x)A_{\gamma^*}(x) \quad (14)$$

for any $\gamma \in \mathbb{G}$. One can find that III is the dual of \emptyset , and, if exists, II is the one of I. We may denote as follows

$$\emptyset = \text{III}^*, \emptyset^* = \text{III} \quad \text{and} \quad \text{I} = \text{II}^*, \text{I}^* = \text{II}.$$

Analogously, we say that ξ_{γ^*} is the dual of ξ_γ .

2.3 Darboux transformations

Now we have a Böchner type operator $\mathcal{L} = A(x)\partial^2 + B(x)\partial = w(x)^{-1}\partial w(x)A(x)\partial$ and its corresponding set Φ of polynomial and quasi-polynomial eigenfunctions such that

$$\mathcal{L}[\phi_{\gamma,n}(x)] = \lambda_{\gamma,n}\phi_{\gamma,n} \quad \text{for all } (\gamma, n) \in \mathbb{G} \times \mathbb{Z}_{\geq 0}. \quad (15)$$

Here and hereafter we set the constant term $C = 0$ on the ground that any Böchner type operator can be rewritten into the form (15) by replacing $\lambda_{\gamma,n} - C \rightarrow \lambda_{\gamma,n}$. Moreover we assume that spectral parameters $\lambda_{\gamma,n}$ are all mutually distinct for simplicity.

Let us take $\phi_{\rho,j}(x) \in \Phi$ to be the seed function of the Darboux transformation, where we call (ρ, j) the *Darboux parameter*. By using this seed function $\phi_{\rho,j}$, the operator \mathcal{L} can be factorized into

$$\mathcal{L} = \mathcal{B} \circ \mathcal{F} + \lambda_{\rho,j}$$

where

$$\begin{aligned} \mathcal{B} &= A(x) (\partial + A'(x)/A(x) - w'(x)/w(x) - \phi'_{\rho,j}(x)/\phi_{\rho,j}(x)) \pi(x)^{-1}, \\ \mathcal{F} &= \pi(x) (\partial - \phi'_{\rho,j}(x)/\phi_{\rho,j}(x)). \end{aligned}$$

Then the Darboux transformed operator and eigenfunctions are respectively given by

$$\widehat{\mathcal{L}} = \mathcal{F} \circ \mathcal{B} + \lambda_{\rho,j}$$

and

$$\widehat{\phi}_{\gamma,n}(x) = \mathcal{T}_{\rho,j}[\phi_{\gamma,n}(x)] = \begin{cases} \mathcal{F}[\phi_{\gamma,n}(x)] = \frac{A_{\rho}(x)}{\xi_{\rho}(x)} \text{Wr}[\phi_{\rho,j}, \phi_{\gamma,n}](x) & \text{if } (\gamma, n) \neq (\rho, j), \\ A_{\rho}(x)/(A(x)w(x)\xi_{\rho}(x)) = \xi_{\rho^*}(x)/A_{\rho^*}(x) & \text{if } (\gamma, n) = (\rho, j), \end{cases}$$

where $\text{Wr}[f_1, f_2](x) = f_1(x)f_2'(x) - f_1'(x)f_2(x)$ and we put the normalization factor $\pi(x) = A_{\rho}(x)\phi_{\rho,j}(x)/\xi_{\rho}(x)$.

It is easy to see that this transformation can be considered as an isospectral transformation of \mathcal{L} and $\phi_{\gamma,n} \in \Phi$:

$$\widehat{\mathcal{L}}[\widehat{\phi}_{\gamma,n}(x)] = \lambda_{\gamma,n} \widehat{\phi}_{\gamma,n}(x) \quad \text{for all } (\gamma, n) \in \mathbb{G} \times \mathbb{Z}_{\geq 0}.$$

Moreover the Darboux transformation \mathcal{T} associated with XOPs preserves the quasi-polynomiality of eigenfunctions, that is,

$$\mathcal{T} : \Phi \rightarrow \widehat{\Phi} = \left\{ \widehat{\phi}_{\gamma,n} = \mathcal{T}_{\rho,j}[\phi_{\gamma,n}] = \widehat{\xi}_{\gamma,n} \widehat{p}_{\gamma,n} \mid (\gamma, n) \in \mathbb{G} \times \mathbb{Z}_{\geq 0} \right\},$$

where $\widehat{p}_{\gamma,n} \in \mathbb{C}[x]$ and

$$\widehat{\xi}_{\gamma,n}(x) = \begin{cases} \xi(x; A, B + \text{Wr}[A_{\rho}, A_{\rho^*}], \gamma) & (\gamma, n) \neq (\rho, j) \\ \xi(x; A, B + \text{Wr}[A_{\rho}, A_{\rho^*}], \gamma^*) & (\gamma, n) = (\rho, j) \end{cases}.$$

By construction the set of all polynomial eigenfunctions transformed with the Darboux parameter (ρ, j) can be presented by

$$\begin{cases} \left\{ \widehat{\phi}_{\gamma,n} \mid (\gamma, n) \in \{\emptyset\} \times \mathbb{Z}_{\geq 0} \setminus \{(\emptyset, j)\} \right\} & \text{if } \rho = \emptyset, \\ \left\{ \widehat{\phi}_{\gamma,n} \mid (\gamma, n) \in \{\emptyset\} \times \mathbb{Z}_{\geq 0} \right\} & \text{if } \rho = \text{I or II}, \\ \left\{ \widehat{\phi}_{\gamma,n} \mid (\gamma, n) \in \{\emptyset\} \times \mathbb{Z}_{\geq 0} \cup \{(\text{III}, j)\} \right\} & \text{if } \rho = \text{III}. \end{cases}$$

3 Recurrence relation for XOPs

In the previous section, we have seen that so-called X_j -OPs of any type can be obtained via one Darboux transformation. In the followings, we denote type 1, 2, 3 X_j -OPs by $\hat{P}_n^{(I,j)}(x)$, $\hat{P}_n^{(II,j)}(x)$, $\hat{P}_n^{(III,j)}(x)$ respectively. Since every X_j -OPs are derived from the ordinary COPs via Darboux transformation, it is straightforward to see that, in the case of when the Darboux parameter is (ρ, j) ,

$$\begin{aligned}\hat{P}_n^{(\rho,j)}(x) &= \hat{\phi}_{\emptyset,n} & (\rho = \text{I, II}), \\ \hat{P}_0^{(III,j)}(x) &= \hat{\phi}_{\text{III},j} \propto 1 \text{ and } \hat{P}_{n+j+1}^{(III,j)}(x) = \hat{\phi}_{\emptyset,n} & (\rho = \text{III}),\end{aligned}$$

for $n \in \mathbb{Z}_{\geq 0}$. Therefore, their explicit form is given as follows:

- (type 1 and type 2 X_j -OPs)

$$\begin{aligned}\hat{P}_n^{(\rho,j)}(x) &= A_{\rho} p_{\rho,j} \left\{ \partial - \frac{\phi'_{\rho,j}}{\phi_{\rho,j}} \right\} P_n(x) = A_{\rho} p_{\rho,j} \left\{ \partial - \left(\frac{\xi'_{\rho}}{\xi_{\rho}} + \frac{p'_{\rho,j}}{p_{\rho,j}} \right) \right\} P_n(x), \quad (\rho = \text{I, II}) \\ \deg(\hat{P}_n^{(\rho,j)}(x)) &= n + j,\end{aligned} \quad (16)$$

- (type 3 X_j -OPs)

$$\begin{aligned}\hat{P}_n^{(III,j)}(x) &= \begin{cases} \frac{\xi_{\emptyset}}{A_{\emptyset}} \propto 1 & (n = 0) \\ A_{\text{III}} p_{\text{III},j} \left\{ \partial - \left(\frac{\xi'_{\text{III}}}{\xi_{\text{III}}} + \frac{p'_{\text{III},j}}{p_{\text{III},j}} \right) \right\} P_{n-j-1}(x) & (n \geq j+1) \\ 0 & (\text{otherwise}) \end{cases} \\ \deg(\hat{P}_n^{(III,j)}(x)) &= n \quad (n = 0, j+1, j+2, \dots).\end{aligned} \quad (17)$$

Let $\mathbb{N}_{\rho,j} = \{n \in \mathbb{Z}_{\geq 0} \mid \hat{P}_n^{(\rho,j)} \neq 0\}$. Their orthogonality relation is given by

$$\int_D \hat{P}_n^{(\rho,j)}(x) \hat{P}_m^{(\rho,j)}(x) \frac{A(x) w(x)}{(A_{\rho}(x) p_{\rho,j}(x))^2} dx = \hat{h}_{nm}^{(\rho,j)}, \quad \rho = \text{I, II, III}, \quad (18)$$

for $n, m \in \mathbb{N}_{\rho,j}$.

We shall proceed the bispectrality of the X_j -OPs, i.e. the recurrence relations. As for the recurrence relations, several results are already discussed. For example, some classical X_j -OPs of type 1 and type 2 are shown to satisfy the following $4j+1$ -recurrence relation [16]:

$$(p_{\rho,j}(x))^2 \hat{P}_n^{(\rho,j)}(x) = \sum_{l=-2j}^{2j} \alpha_{n,l} \hat{P}_{n+l}^{(\rho,j)}(x), \quad (\rho = \text{I, II}). \quad (19)$$

We have found there exists another recurrence relation, which might be the simplest recurrence formula for XOPs.

Theorem 3.1. *For any $\rho \in \{\text{I, II, III}\}$ and $n \in \mathbb{N}_{\rho,j}$, X_j -OPs $\{\hat{P}_n^{(\rho,j)}\}_{n=0}^{\infty}$ satisfy the following $2j+3$ -recurrence relation:*

$$I_j(x) \hat{P}_n^{(\rho,j)}(x) = \sum_{l=-j-1}^{j+1} \beta_{n,l} \hat{P}_{n+l}^{(\rho,j)}(x), \quad (20)$$

with $\widehat{P}_{n<0}^{(\rho,j)} \equiv 0$ and

$$I_j(x) = \int p_{\rho,j}(x) dx.$$

Remark. In case $j = 1$, this gives the same recurrence relation as in [16]. In [7, 13], X_j -OPs are shown to satisfy 5-term recurrence relation whose coefficients take polynomial forms in x , while all the coefficients $\beta_{n,l}$ in (20) are constants.

Before the proof, we introduce the following lemma concerning type 3 XOPs.

Lemma 3.2. *There exist some constants α, β s.t.*

$$I_j(x) = \alpha \widehat{P}_{j+1}^{(\text{III},j)}(x) + \beta \widehat{P}_0^{(\text{III},j)}(x).$$

Proof. Since $\widehat{P}_0^{(\text{III},j)}(x) \propto 1$, it is sufficient to prove

$$I'_j(x) = p_{\text{III},j}(x) \propto (\widehat{P}_{j+1}^{(\text{III},j)}(x))'.$$

From the definition (17) and $w(x)\xi_{\text{III}}(x) = 1$, we have

$$\begin{aligned} (\widehat{P}_{j+1}^{(\text{III},j)}(x))' &= \frac{d}{dx} \left(A_{\text{III}}(x) p_{\text{III},j}(x) \left\{ \frac{d}{dx} - \left(\frac{\xi'_{\text{III}}(x)}{\xi_{\text{III}}(x)} + \frac{p'_{\text{III},j}(x)}{p_{\text{III},j}(x)} \right) \right\} P_0(x) \right) \\ &\propto \frac{d}{dx} \left(A_{\text{III}}(x) p_{\text{III},j}(x) \frac{\phi'_{\text{III},j}(x)}{\phi_{\text{III},j}(x)} \right) \\ &= \frac{d}{dx} (A_{\text{III}}(x) w(x) \phi'_{\text{III},j}(x)). \end{aligned}$$

Recall from lemma 2.1 $A_{\text{III}}(x) = A(x)$ and the equivalent form of the Böchner type operator (2), we see as a consequence

$$(\widehat{P}_{j+1}^{(\text{III},j)}(x))' \propto \lambda_{\text{III},j} w(x) \phi_{\text{III},j}(x) = \lambda_{\text{III},j} p_{\text{III},j}(x).$$

This completes the proof. □

Proof of Theorem 3.1. Due to the completeness of the XOPs, it is possible to write

$$I_j(x) \widehat{P}_n^{(\rho,j)}(x) = \sum_{l=0}^{\infty} \varepsilon_{n,l} \widehat{P}_l^{(\rho,j)}(x)$$

uniquely. From the above expansion, we only have to prove

$$\varepsilon_{n,l} = 0, \quad (|l - n| > j + 1).$$

Using the orthogonality relation (18), one can find

$$\varepsilon_{n,l} = (\widehat{h}_l^{(\rho,j)})^{-1} \int_D I_j(x) \widehat{P}_n^{(\rho,j)}(x) \widehat{P}_l^{(\rho,j)}(x) \frac{A(x)w(x)}{(A_\rho(x)p_{\rho,j}(x))^2} dx.$$

In the followings, we shall examine the quantity ε_l for each case.

- ($\rho = \text{I, II}$) From the definition (16), we have

$$\begin{aligned}\widehat{h}_l^{(\rho,j)} \varepsilon_{n,l} &= \int_D I_j \left(P'_n - \frac{\phi'_{\rho,j}}{\phi_{\rho,j}} P_n \right) \left(P'_l - \frac{\phi'_{\rho,j}}{\phi_{\rho,j}} P_l \right) A w dx \\ &= \int_D I_j P'_n P'_l A w - \int_D \partial(P_l P_n) \frac{\phi'_{\rho,j}}{\phi_{\rho,j}} I_j A w dx + \int_D P_l P_n \left(\frac{\phi'_{\rho,j}}{\phi_{\rho,j}} \right)^2 I_j A w dx.\end{aligned}$$

From the property of COPs (6), recalling I_j is the polynomial in x of $j+1$ degree, it is easy to see

$$\int_D I_j P'_n P'_l A w dx = 0 \quad (|n-l| > j+1).$$

Then for $|l-n| > j+1$, we have performing the partial integral

$$\begin{aligned}\widehat{h}_l^{(\rho,j)} \varepsilon_{n,l} &= - \int_D \partial(P_l P_n) \frac{\phi'_{\rho,j}}{\phi_{\rho,j}} I_j A w dx + \int_D P_l P_n \left(\frac{\phi'_{\rho,j}}{\phi_{\rho,j}} \right)^2 I_j A w dx \\ &= \int_D P_l P_n \partial \left(\frac{\phi'_{\rho,j}}{\phi_{\rho,j}} I_j A w \right) dx + \int_D P_l P_n \left(\frac{\phi'_{\rho,j}}{\phi_{\rho,j}} \right)^2 I_j A w dx \\ &= \int_D P_l P_n \left(\frac{\phi''_{\rho,j}}{\phi_{\rho,j}} I_j A w + \frac{\phi'_{\rho,j}}{\phi_{\rho,j}} p_{\rho,j} A w + \frac{\phi'_{\rho,j}}{\phi_{\rho,j}} I_j B w \right) dx \\ &= \int_D P_l P_n I_j \frac{A \phi''_{\rho,j} + B \phi'_{\rho,j} w}{\phi_{\rho,j}} dx + \int_D P_l P_n \left(\frac{\xi'_\rho}{\xi_\rho} + \frac{p'_{\rho,j}}{p_{\rho,j}} \right) A p_{\rho,j} w dx.\end{aligned}$$

From (15), we have for $|l-n| > j$,

$$\widehat{h}_l^{(\rho,j)} \varepsilon_{n,l} = \int_D P_l P_n (\lambda_{\rho,j} I_j + A p'_{\rho,j} + \eta_\rho A p_{\rho,j}) w dx.$$

From the lemma 2.1, one can verify that $A \eta_\rho$ is a polynomial in x of degree up to 1. Then, $\deg(\lambda_{\rho,j} I_j + A p'_{\rho,j} + \eta_\rho A p_{\rho,j}) = j+1$, which amounts to

$$\varepsilon_{n,l} = 0, \quad |n-l| > j+1.$$

- ($\rho = \text{III}$) When $n = 0$, from the lemma 3.2, we directly have

$$I_j(x) \widehat{P}_0^{(\text{III},j)}(x) = \alpha \widehat{P}_{j+1}^{(\text{III},j)}(x) + \beta \widehat{P}_0^{(\text{III},j)}(x), \quad (\exists \alpha, \beta \in \mathbb{C}). \quad (21)$$

This is nothing but the $2j+3$ -recurrence relation. For $n \geq j+1$, in the fashion similar to the case $\rho = \text{I, II}$, one can find

$$\varepsilon_{n,l} = 0, \quad |n-l| > j+1 \quad \text{for} \quad l \geq j+1.$$

Furthermore, from (21), it is straightforward to see from the orthogonality relation (18)

$$\begin{aligned}\varepsilon_{n,0} &= (\widehat{h}_0^{(\rho,j)})^{-1} \int_D I_j \widehat{P}_n^{(\rho,j)} \widehat{P}_0^{(\rho,j)} \frac{A w}{(A_\rho p_{\rho,j})^2} dx \\ &= (\widehat{h}_0^{(\rho,j)})^{-1} \int_D \widehat{P}_n^{(\rho,j)} (\alpha \widehat{P}_{j+1}^{(\rho,j)} + \beta \widehat{P}_0^{(\rho,j)}) \frac{A w}{(A_\rho p_{\rho,j})^2} dx \\ &= 0 \quad (n > j+1).\end{aligned}$$

Therefore, for $n \geq 1$, we have as a consequence

$$\varepsilon_{n,l} = 0, \quad |n - l| > j + 1.$$

Combining these results, we have the $2j + 3$ -recurrence relations for every type X_j -OPs. This completes the proof. \square

It is worth noting that from the proof we can evaluate all the coefficients of the recurrence relation in terms of the orthogonality constant with respect to the ordinary COPs. We give the collection of the explicit expression of the recurrence relation for some classical XOPs in the following section. It should also be remarked if we take $j = 0$, which corresponds to the ordinary OPs, the recurrence relation (20) becomes the three term recurrence relation as ordinary OPs satisfy. In that sense, the recurrence relation (20) is a natural generalization of the three term recurrence relation for OPs.

4 Recurrence formulas for the classical XOPs

In this section, as is mentioned in the previous section, we give the explicit form of the recurrence relation for the classical X_j -OPs including Hermite, Laguerre and Jacobi polynomials. In order to avoid the complexity of the recurrence relation, we especially fix all polynomials (including OPs, XOPs) to monic ones.

The ordinary Hermite, Laguerre and Jacobi polynomials in monic form are defined by the Rodrigues' formula[11]:

$$\begin{aligned} H_n(x) &= \frac{(-1)^n}{2^n} e^{x^2} \left(\frac{d}{dx} \right)^n e^{-x^2}, \\ L_n^{(a)}(x) &= (-1)^n \frac{e^x}{x^a} \left(\frac{d}{dx} \right)^n \left[\frac{x^{n+a}}{e^x} \right], \\ J_n^{(a,b)}(x) &= \frac{(-1)^n}{(n+a+b+1)_n} (1-x)^{-a} (1+x)^{-b} \left(\frac{d}{dx} \right)^n [(1-x)^{n+a} (1+x)^{n+b}], \end{aligned}$$

respectively, where $(a)_n = a(a+1) \cdots (a+n-1)$ is the standard Pochhammer symbol. In the followings, we give the corresponding recurrence formula (20) in each case.

X_j -Hermite polynomials

1. There are no type-1 exceptional Hermite polynomials.
2. There are no type-2 exceptional Hermite polynomials.
3. Type-3 X_j -Hermite polynomials are introduced as follows ($j = 2, 4, 6, \dots$):

$$\hat{H}_n^{(\text{III},j)}(x) = \begin{cases} 1 & (n = 0) \\ -2^{-1}i^{-j} (H_j(ix)(\partial - 2x)H_{n-j-1}(x) - \partial(H_j(ix))H_{n-j-1}(x)) & (n \geq j+1) \\ 0 & (\text{otherwise}) \end{cases}$$

with $i = \sqrt{-1}$ and the polynomial part of the seed function and decoupling factor chosen

$$\phi_{\text{III},j}(x) = e^{x^2} H_j(ix), \quad \pi(x) = H_j(ix).$$

It is easy to see

$$\int p_{\text{III},j}(x) dx = \frac{-i}{j+1} H_{j+1}(ix) + C,$$

which will give us $2j+3$ -recurrence relations:

- $j = 2$: 7-term recurrence relation ($n \neq 1, 2$)

$$\begin{aligned} iH_3(ix) \widehat{H}_n^{(\text{III},2)}(x) &= \widehat{H}_{n+3}^{(\text{III},2)}(x) + \frac{3}{2} n \widehat{H}_{n+1}^{(\text{III},2)}(x) + \frac{3}{4} n(n-3) \widehat{H}_{n-1}^{(\text{III},2)}(x) \\ &\quad + \frac{1}{8} n(n-4)(n-5) \widehat{H}_{n-3}^{(\text{III},2)}(x) \end{aligned}$$

$$\text{with } iH_3(ix) = x^3 + 3x/2.$$

X_j -Laguerre polynomials

1. Type-1 X_j -Laguerre polynomials are introduced as follows:

$$\widehat{L}_n^{(\text{I},j)}(x; a) = (-1)^{j+1} \left(L_j^{(a)}(-x)(\partial - 1)L_n^{(a)}(x) - \partial(L_j^{(a)}(-x))L_n^{(a)}(x) \right),$$

where the polynomial part of the seed function and decoupling factor are chosen

$$\phi_{\text{I},j}(x) = e^x L_j^{(a)}(-x), \quad \pi(x) = L_j^{(a)}(-x),$$

respectively. It is easy to verify

$$\int p_{\text{I},j}(x) dx = -\frac{1}{j+1} L_{j+1}^{(a)}(-x) + C,$$

which will give us $2j+3$ -recurrence relations:

- $j = 1$: 5-term recurrence relation

$$\begin{aligned} L_2^{(a-1)}(-x) \widehat{L}_n^{(\text{I},1)}(x; a) &= \widehat{L}_{n+2}^{(\text{I},1)}(x; a) + 4(n+a+2) \widehat{L}_{n+1}^{(\text{I},1)}(x; a) \\ &\quad + 2(n+a+2)(3n+2a+2) \widehat{L}_n^{(\text{I},1)}(x; a) + 4(n+a+2)(n+a) n \widehat{L}_{n-1}^{(\text{I},1)}(x; a) \\ &\quad + (n+a+2)(n+a-1) n(n-1) \widehat{L}_{n-2}^{(\text{I},1)}(x; a) \end{aligned}$$

$$\text{with } L_2^{(a-1)}(-x) = x^2 + (2a+2)x + a(a+1),$$

- $j = 2$: 7-term recurrence relation

$$\begin{aligned}
& -L_3^{(a-1)}(-x)\widehat{L}_n^{(I,2)}(x;a) \\
& = \widehat{L}_{n+3}^{(I,2)}(x;a) + 6(n+a+3)\widehat{L}_{n+2}^{(I,2)}(x;a) + 3(n+a+3)(5n+4a+8)\widehat{L}_{n+1}^{(I,2)}(x;a) \\
& \quad + 4(n+a+3)(n+a+1)(5n+2a+4)\widehat{L}_n^{(I,2)}(x;a) + 3(n+a+3)(n+a)(5n+4a+3)n\widehat{L}_{n-1}^{(I,2)}(x;a) \\
& \quad + 6(n+a+3)(n+a-1)_2(n-1)_2\widehat{L}_{n-2}^{(I,2)}(x;a) + (n+a+3)(n+a-2)_2(n-2)_3\widehat{L}_{n-3}^{(I,2)}(x;a)
\end{aligned}$$

$$\text{with } -L_3^{(a-1)}(-x) = x^3 + 3(a+2)x^2 + 3(a+1)(a+2)x + a(a+1)(a+2).$$

2. Type-2 X_j -Laguerre polynomials are introduced as follows:

$$\widehat{L}_n^{(II,j)}(x;a) = (n+a-j)^{-1} \left(L_j^{(-a)}(x)(x\partial + a)L_n^{(a)}(x) - x\partial(L_j^{(-a)}(x))L_n^{(a)}(x) \right),$$

where the polynomial part of the seed function and decoupling factor are chosen

$$\phi_{II,j}(x) = x^{-a}L_j^{(-a)}(x), \quad \pi(x) = xL_j^{(-a)}(x),$$

respectively. It is easy to verify

$$\int p_{II,j}(x)dx = \frac{1}{j+1}L_{j+1}^{(-a-1)}(x) + C,$$

which will give us $2j+3$ -recurrence relations:

- $j = 1$: 5-term recurrence relation

$$\begin{aligned}
& L_2^{(-a-1)}(x)\widehat{L}_n^{(II,1)}(x;a) \\
& = \widehat{L}_{n+2}^{(II,1)}(x;a) + 4(n+a)\widehat{L}_{n+1}^{(II,1)}(x;a) + 2(n+a-1)(3n+2a+1)\widehat{L}_n^{(II,1)}(x;a) \\
& \quad + 4(n+a-2)(n+a)n\widehat{L}_{n-1}^{(II,1)}(x;a) + (n+a-3)(n+a)n(n-1)\widehat{L}_{n-2}^{(II,1)}(x;a)
\end{aligned}$$

$$\text{where } L_2^{(-a-1)}(x) = x^2 + (2a-2)x + a(a-1),$$

- $j = 2$: 7-term recurrence relation

$$\begin{aligned}
& L_3^{(-a-1)}(x)\widehat{L}_n^{(II,2)}(x;a) \\
& = \widehat{L}_{n+3}^{(II,2)}(x;a) + 6(n+a)\widehat{L}_{n+2}^{(II,2)}(x;a) + 3(n+a-1)(5n+4a+2)\widehat{L}_{n+1}^{(II,2)}(x;a) \\
& \quad + 4(n+a-2)(n+a)(5n+2a+1)\widehat{L}_n^{(II,2)}(x;a) + 3(n+a-3)(n+a)(5n+4a-3)n\widehat{L}_{n-1}^{(II,2)}(x;a) \\
& \quad + 6(n+a-4)(n+a-1)_2(n-1)_2\widehat{L}_{n-2}^{(II,2)}(x;a) + (n+a-5)(n+a-1)_2(n-2)_3\widehat{L}_{n-3}^{(II,2)}(x;a)
\end{aligned}$$

$$\text{where } L_3^{(-a-1)}(x) = x^3 + 3(a-2)x^2 + 3(a-1)(a-2)x + a(a-1)(a-2).$$

3. Type-3 X_j -Laguerre polynomials are introduced as follows:

$$\widehat{L}_n^{(\text{III},j)}(x; a) = \begin{cases} 1 & (n = 0) \\ (-1)^{j+1} \left(L_j^{(-a)}(-x)(x\partial + a - x)L_{n-j-1}^{(a)}(x) - x\partial(L_j^{(-a)}(-x))L_{n-j-1}^{(a)}(x) \right) & (n \geq j+1) \\ 0 & (\text{otherwise}) \end{cases},$$

where the polynomial part of the seed function and decoupling factor are chosen

$$\phi_{\text{III},j}(x) = x^{-a}e^x L_j^{(-a)}(-x), \quad \pi(x) = xL_j^{(-a)}(-x),$$

respectively. It is easy to verify

$$\int p_{\text{III},j}(x)dx = -\frac{1}{j+1}L_{j+1}^{(-a-1)}(-x) + C,$$

which will give us $2j+3$ -recurrence relations:

- $j = 1$: 5-term recurrence relation ($n \neq 1$)

$$\begin{aligned} & L_2^{(-a-1)}(-x) \widehat{L}_n^{(\text{III},1)}(x; a) \\ &= \widehat{L}_{n+2}^{(\text{III},1)}(x; a) + 4n \widehat{L}_{n+1}^{(\text{III},1)}(x; a) + 2n(3n + a - 4) \widehat{L}_n^{(\text{III},1)}(x; a) \\ & \quad + 4n(n-2)(n+a-2) \widehat{L}_{n-1}^{(\text{III},1)}(x; a) + n(n-3)(n+a-2)(n+a-3) \widehat{L}_{n-2}^{(\text{III},1)}(x; a) \end{aligned}$$

$$\text{where } L_2^{(-a-1)}(-x) = x^2 + (-2a+2)x + a(a-1),$$

- $j = 2$: 7-term recurrence relation ($n \neq 1, 2$)

$$\begin{aligned} & -L_3^{(-a-1)}(-x) \widehat{L}_n^{(\text{III},2)}(x; a) \\ &= \widehat{L}_{n+3}^{(\text{III},2)}(x; a) + 6n \widehat{L}_{n+2}^{(\text{III},2)}(x; a) + 3n(5n + a - 7) \widehat{L}_{n+1}^{(\text{III},2)}(x; a) \\ & \quad + 4n(n-2)(5n+3a-11) \widehat{L}_n^{(\text{III},2)}(x; a) + 3n(n-3)(n+a-3)(5n+a-12) \widehat{L}_{n-1}^{(\text{III},2)}(x; a) \\ & \quad + 6n(n-4)_2(n+a-4)_2 \widehat{L}_{n-2}^{(\text{III},2)}(x; a) + n(n-5)_2(n+a-5)_3 \widehat{L}_{n-3}^{(\text{III},2)}(x; a) \end{aligned}$$

$$\text{where } -L_3^{(-a-1)}(-x) = x^3 - 3(a-2)x^2 + 3(a-1)(a-2)x - a(a-1)(a-2).$$

X_j -Jacobi polynomials As for X_j -Jacobi polynomials, the corresponding recurrence relations take simple form if $a = b$. We thus restrict the case to $a = b$.

1. Type-1 X_j -Jacobi polynomials are introduced as follows:

$$\widehat{J}_n^{(\text{I},1)}(x; a, b) = \frac{J_j^{(a,-b)}(x)((1+x)\partial + b)J_n^{(a,b)}(x) - (1+x)\partial(J_j^{(a,-b)}(x))J_n^{(a,b)}(x)}{n+b-j}$$

where the polynomial part of the seed function and decoupling factor are chosen

$$\phi_{\text{I},j}(x) = (1+x)^{-b}J_j^{(a,-b)}(x), \quad \pi(x) = (1+x)J_j^{(a,-b)}(x),$$

respectively. It is easy to verify

$$\int p_{\text{I},j}(x)dx = \frac{1}{j+1}J_{j+1}^{(a-1,-b-1)}(x) + C,$$

which will give us $2j+3$ -recurrence relations:

- $j = 1$ ($a = b$): 5-term recurrence relation

$$J_2^{(a-1, -a-1)}(x) \widehat{J}_n^{(I,1)}(x; a, a) = \widehat{J}_{n+2}^{(I,1)}(x; a, a) + 2a \widehat{J}_{n+1}^{(I,1)}(x; a, a) \\ + \frac{2(4a^2 - 1)(n + a - 1)(n + a + 2)}{(2n + 2a - 1)(2n + 2a + 3)} \widehat{J}_n^{(I,1)}(x; a, a) + \frac{8a((n + a)^2 - 4)n(n + 2a)}{(2n + 2a - 2)_2(2n + 2a + 1)_2} \widehat{J}_{n-1}^{(I,1)}(x; a, a) \\ + \frac{4(n + a - 3)(n + a + 2)(n - 1)_2(n + 2a - 1)_2}{(2n + 2a - 3)_3(2n + 2a - 1)_3} \widehat{J}_{n-2}^{(I,1)}(x; a, a)$$

where $J_2^{(a-1, -a-1)}(x) = x^2 + 2ax + 2a^2 - 1$,

- $j = 2$ ($a = b$): 7-term recurrence relation

$$J_3^{(a-1, -a-1)}(x) \widehat{J}_n^{(I,2)}(x; a, a) = \widehat{J}_{n+3}^{(I,2)}(x; a, a) + \frac{3a}{2} \widehat{J}_{n+2}^{(I,2)}(x; a, a) \\ + \frac{(4a^2 - 1)(n + a + 3)(n + a - 1)}{(2n + 2a - 1)(2n + 2a + 5)} \widehat{J}_{n+1}^{(I,2)}(x; a, a) \\ + \frac{a(4a^2 - 1)(n + a + 3)(n + a - 2)}{3(2n + 2a - 1)(2n + 2a + 3)} \widehat{J}_n^{(I,2)}(x; a, a) \\ + \frac{(4a^2 - 1)(n + a + 3)(n + a - 3)}{(2n + 2a - 1)(2n + 2a + 2)} \frac{n(n + 2a)}{(2n + 2a - 3)(2n + 2a + 3)} \widehat{J}_{n-1}^{(I,2)}(x; a, a) \\ + \frac{6a(n + a + 3)(n + a - 4)}{(2n + 2a - 1)^2} \frac{(n - 1)_2(n + 2a - 1)_2}{(2n + 2a - 4)_2(2n + 2a + 1)_2} \widehat{J}_{n-2}^{(I,2)}(x; a, a) \\ + \frac{4(n + a + 3)(n + a - 5)}{(2n + 2a - 1)(2n + 2a - 3)} \frac{(n - 2)_3(n + 2a - 2)_3}{(2n + 2a - 5)_3(2n + 2a - 1)_3} \widehat{J}_{n-3}^{(I,2)}(x; a, a)$$

where $J_3^{(a-1, -a-1)}(x) = x^3 + 3ax^2/2 + (a - 1)(a + 1)x + a(2a^2 - 5)/6$,

2. Type-2 X_j -Jacobi polynomials are introduced as follows:

$$\widehat{J}_n^{(II,j)}(x; a, b) = \frac{J_j^{(-a,b)}(x)((1 - x)\partial - a)J_n^{(a,b)}(x) - (1 - x)\partial(J_j^{(-a,b)}(x))J_n^{(a,b)}(x)}{-n - a + j}$$

where the polynomial part of the seed function and decoupling factor are chosen

$$\phi_{II,j}(x) = (1 - x)^{-a} J_j^{(-a,b)}(x), \quad \pi(x) = (1 - x) J_j^{(-a,b)}(x),$$

respectively. In this case the recurrence relations can be immediately derived by observing

$$\widehat{J}_n^{(II,j)}(x; a, a) = (-1)^{n+j} \widehat{J}_n^{(I,j)}(-x; a, a).$$

3. Type-3 X_j -Jacobi polynomials are introduced as follows:

$$\widehat{J}_n^{(III,j)}(x; a, b) = \begin{cases} 1 & (n = 0) \\ (J_j^{(-a,-b)}(x)((x^2 - 1)\partial + (a + b)x + a - b)J_{n-j-1}^{(a,b)}(x) \\ + (1 - x^2)\partial(J_j^{(-a,-b)}(x))J_{n-j-1}^{(a,b)}(x))/(n + a + b - 2j - 1) & (n \geq j + 1) \\ 0 & (\text{otherwise}) \end{cases}$$

where the polynomial part of the seed function and decoupling factor are chosen

$$\phi_{\text{III},j}(x) = (1-x)^{-a}(1+x)^{-b}J_j^{(-a,-b)}(x), \quad \pi(x) = (1-x)^2J_j^{(-a,-b)}(x),$$

respectively. It is easy to verify

$$\int p_{\text{III},j}(x)dx = \frac{1}{j+1}J_{j+1}^{(-a-1,-b-1)}(x) + C,$$

which will give us $2j+3$ -recurrence relations:

- $j = 1$ ($a = b$): 5-term recurrence relation ($n \neq 1$)

$$\begin{aligned} J_2^{(-a-1,-a-1)}(x) \hat{J}_n^{(\text{III},1)}(x; a, a) &= \hat{J}_{n+2}^{(\text{III},1)}(x; a, a) \\ &+ \frac{2(2a+1)n(n+2a-3)}{(2a-1)(2n+2a-5)(2n+2a-1)} \hat{J}_n^{(\text{III},1)}(x; a, a) \\ &+ \frac{n(n-3)(n+2a-5)(n+2a-2)}{8(2n+2a-5)(n+a-7/2)_3} \hat{J}_{n-2}^{(\text{III},1)}(x; a, a) \end{aligned}$$

where $J_2^{(-a-1,-a-1)}(x) = x^2 + 1/(2a-1)$,

- $j = 2$ ($a = b$): 7-term recurrence relation ($n \neq 1, 2$)

$$\begin{aligned} J_3^{(-a-1,-a-1)}(x) \hat{J}_n^{(\text{III},2)}(x; a, a) &= \hat{J}_{n+3}^{(\text{III},2)}(x; a, a) \\ &+ \frac{3(2a+1)n(n+2a-4)}{(2a-3)(2n+2a-1)(2n+2a-7)} \hat{J}_{n+1}^{(\text{III},2)}(x; a, a) \\ &+ \frac{3(2a+1)n(n-3)(n+2a-3)(n+2a-6)}{2^4(2a-3)(n+a-9/2)_4} \hat{J}_{n-1}^{(\text{III},2)}(x; a, a) \\ &+ \frac{n(n-4)(n-5)(n+2a-8)(n+2a-4)_2}{2^6(n+a-9/2)_2(n+a-11/2)_4} \hat{J}_{n-3}^{(\text{III},2)}(x; a, a) \end{aligned}$$

where $J_3^{(-a-1,-a-1)}(x) = x^3 + 3x/(2a-3)$.

5 Concluding Remarks

In this paper, we have given a brief review of XOPs especially focusing on possible quasi-polynomial solutions to the B"ochner type operator and the corresponding Darboux transformations, from which X_j -OPs of all kinds are obtained. Observing the structure of the Darboux transformation carefully, we have shown a new recurrence formula for X_j -OPs. We have also given the explicit form of the recurrence relation for several classical X_j -OPs.

The recurrence formula holds for every type X_j -OPs and all the coefficients except spectra of the recurrence formula are constants. This means that X_j -OPs diagonalize some band-matrices which can be regarded as the generalized Jacobi matrix, which we believe produces the exactly solvable physical model. Recently, multi-indexed OPs are derived from the multi-step Darboux-transformations and the recurrence formula with constant coefficients for multi-indexed OPs are expected to be obtained by our strategy[15, 10]. In [5], the higher-order recurrence relation is given from the "dual" aspects of OPs with higher order difference equation, which is definitely related to our recurrence formula. We shall plan to return to these in the future publications.

Acknowledgements

The authors thank to A. Duran, R. Milson, S. Odake, R. Sasaki, L. Vinet and A. Zhedanov for fruitful discussions and helpful advice. They also give special thank to D. Gómez-Ullate, F. Marcellán and M. Rodríguez for organizing the workshop on “XOPs and exact solutions in mathematical physics” at Segovia, Spain. The research of ST was supported in part by JSPS KAKENHI Grant Numbers 25400110.

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